

On volumes of quaternionic hyperbolic n -orbifolds

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Abstract: By use of H. C. Wang's bound on the radius of a ball embedded in the fundamental domain of a lattice of a semisimple Lie group, we construct an explicit lower bound for the volume of a quaternionic hyperbolic orbifold that depends only on dimension.

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1 Introduction

Let \mathbb{F} be real numbers \mathbb{R} , complex numbers \mathbb{C} or quaternions \mathbb{H} , and $\mathbf{H}_{\mathbb{F}}^n$ the n -dimensional hyperbolic space over \mathbb{F} . Let G be the linear groups which act as the isometries in $\mathbf{H}_{\mathbb{F}}^n$. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and \mathbb{H} , G are usually denoted by $O(n, 1)$, $U(n, 1)$ and $Sp(n, 1)$, respectively. A hyperbolic orbifold \mathcal{Q} is a quotient of $\mathbf{H}_{\mathbb{F}}^n$ by a discrete subgroup Γ of G . An orbifold $\mathbf{H}_{\mathbb{F}}^n/\Gamma$ is a manifold when Γ contains no elements of finite order.

In 1945, Siegel [18, 19] posed the problem of identifying the minimal covolume lattices of isometries of real hyperbolic n -space, or more generally rank-1 symmetric spaces. He solved the problem in two dimensions and Gehring and Martin [10] solved similar problem in three dimensions. For the general cases, the results of Wang and Gunther [9, 20, 21] imply that the hyperbolic volumes of \mathcal{Q} form a discrete subset of \mathbb{R} .

Firedland and Hersonsky [7] constructed a lower bound for r_n , the largest number such that every hyperbolic n -manifold contains a round ball of that radius. From this one can compute an explicit lower bound for the volume of a hyperbolic n -manifold. Recently, some analogous results have been obtained in complex and quaternionic settings [5, 22]

Adeboye obtained an explicit lower bound for the volume of a real hyperbolic orbifold depending on the dimension [1]. The main tool is the spectral radius of the involved matrices. Such a technique was employed latter in complex and quaternionic settings [8, 12].

Recently Adeboye and Wei reconsidered the question of lower bound for the volume of a real hyperbolic orbifold with the tools of Lie group and Riemannian submersion [2]. They obtained the following theorem.

Theorem 1.1. (Theorem 0.1 of Adeboye and Wei [2]) *The volume of a real hyperbolic n -orbifold is bounded below by $\mathcal{R}(n)$, an explicit constant depending only on dimension, given by*

$$\mathcal{R}(n) = \frac{2^{\frac{6-n}{4}} \pi^{\frac{n}{4}} (n-2)!(n-4)! \cdots 1}{(2+9n)^{\frac{n^2+n}{4}} \Gamma(\frac{n^2+n}{4})} \int_0^{\min[0.0806\sqrt{2+9n}, \pi]} (\sin \rho)^{\frac{n^2+n-2}{2}} d\rho.$$

Such work significantly improved upon the volume bounds of [1, 15]. The authors also obtained the following result in complex setting.

Theorem 1.2. (Theorem 0.1 of Adeboye and Wei [3]) *The volume of a complex hyperbolic n -orbifold is bounded below by $\mathcal{C}(n)$, an explicit constant depending only on dimension, given by*

$$\mathcal{C}(n) = \frac{2^{n^2+n+1} \pi^{\frac{n}{2}} (n-1)!(n-2)! \cdots 3!2!1!}{(36n+21)^{\frac{n^2+2n}{2}} \Gamma(\frac{n^2+2n}{2})} \int_0^{\min[0.06925\sqrt{36n+21}, \pi]} (\sin \rho)^{n^2+2n-1} d\rho.$$

As interest in quaternionic hyperbolic space has grown, many results from real and complex hyperbolic geometry have been carried over to the quaternionic arena (see [4, 5, 14] et al). Due to the noncommutativity of quaternions, the analogous problems in quaternionic setting are sometimes more complicated.

Motivated by the ideas of Adeboye and Wei in [2, 3], we will consider the question of lower bound for the volume of a quaternionic hyperbolic orbifold with the tools of Lie group and Riemannian submersion.

We will construct a Riemannian submersion from the quotient $\mathrm{Sp}(n, 1)/\Gamma$ to the quotient $\mathbf{H}_{\mathbb{H}}^n/\Gamma$. With this Riemannian submersion, we can employ Wang's result [20, Theorem 5.2] to produce an inscribed ball of radius $\frac{R_{\mathrm{Sp}(n, 1)}}{2}$ in $\mathbf{H}_{\mathbb{H}}^n/\Gamma$ and obtain the lower bound by a comparison theorem of Gunther [9, Theorem 3.101].

Our main result is the following theorem.

Theorem 1.3. *The volume of a quaternionic hyperbolic n -orbifold is bounded below by $\mathcal{Q}(n)$, an explicit constant depending only on dimension, given by*

$$\mathcal{Q}(n) = \frac{\pi^{\frac{3n}{2}} (2n+1)!(2n-1)! \cdots 5!3!1!}{2^{n-1} \Gamma(\frac{2n^2+5n+3}{2}) \Gamma(\frac{4n+1}{2}) (\frac{3+4\sqrt{2}}{2})^{\frac{2n^2+5n+3}{2}}} \int_0^{0.2372} (\sin \rho)^{2n^2+5n+2} d\rho.$$

As in [2, 3], the volume bounds for hyperbolic orbifolds provide information on the order of the symmetry groups of hyperbolic manifolds. Following Hurwitz's formula for groups acting on surfaces, we have the following corollary.

Corollary 1.1. *Let M be a quaternionic hyperbolic n -manifold. Let H be a group of isometries of M . Then*

$$|H| \leq \frac{\mathrm{Vol}(M)}{\mathcal{Q}(n)}.$$

The paper is organized as follows. Section 2 contains some necessary background material for quaternionic hyperbolic geometry. In Section 3, we will present the Cartan decomposition of $\mathfrak{sp}(n, 1)$ and obtain the standard \mathbb{R} -vector space basis for it. Also we will define the canonical metric in Lie group $\mathrm{Sp}(n, 1)$. Section 4 aims to obtain the bound of the sectional curvatures of $\mathrm{Sp}(n, 1)$ with respect to the scaled canonical metric. In order to obtain better estimate, we will use some formulae of the connection and curvature which are slight different from those in [3]. Section 5 contains the proof of Theorem 1.3 with the similar route map employed in [3]. In Section 6, we reestimate the bound of the sectional curvatures of $\mathrm{SO}_o(n, 1)$ and $\mathrm{SU}(n, 1)$. These new bounds imply slight improvements of the results in [2, 3].

2 Quaternionic hyperbolic space

In this section, we give some necessary background materials of quaternionic hyperbolic geometry. More details can be found in [6, 14, 16].

We recall that a real quaternion is of the form $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$ where $q_i \in \mathbb{R}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Let $\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$ and $|q| = \sqrt{\bar{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ be the conjugate and modulus of q , respectively.

Let $\mathbb{H}^{n,1}$ be the vector space of dimension $n+1$ over \mathbb{H} with the unitary structure defined by the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z} = \overline{w_1} z_1 + \cdots + \overline{w_n} z_n - \overline{w_{n+1}} z_{n+1},$$

where \mathbf{z} and \mathbf{w} are the column vectors in $\mathbb{H}^{n,1}$ with entries (z_1, \dots, z_{n+1}) and (w_1, \dots, w_{n+1}) respectively, \cdot^* denotes the conjugate transpose and J is the Hermitian matrix

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

We define a unitary transformation g to be an automorphism of $\mathbb{H}^{n,1}$, that is, a linear bijection such that $\langle g(\mathbf{z}), g(\mathbf{w}) \rangle = \langle \mathbf{z}, \mathbf{w} \rangle$ for all \mathbf{z} and \mathbf{w} in $\mathbb{H}^{n,1}$. We denote the group of all unitary transformations by $\mathrm{Sp}(n, 1)$, which is the noncompact Lie group

$$\mathrm{Sp}(n, 1) = \{A \in \mathrm{GL}(n+1, \mathbb{H}) : A^* J A = J\}. \quad (1)$$

Let

$$V_- = \{\mathbf{z} \in \mathbb{H}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}.$$

It is obvious that V_- is invariant under $\mathrm{Sp}(n, 1)$. Let

$$\mathbb{P} : \mathbb{H}^{n,1} - \{0\} \rightarrow \mathbb{H}\mathbf{P}^n$$

be the canonical projection onto quaternionic projective space. Quaternionic hyperbolic n -space, $\mathbf{H}_{\mathbb{H}}^n$, is defined to be the space $\mathbb{P}(V_-)$ together the Bergman metric. The Bergman metric on $\mathbf{H}_{\mathbb{H}}^n$ is given by the distance formula

$$\cosh^2 \frac{\rho(z, w)}{2} = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}, \quad \text{where } \mathbf{z} \in \mathbb{P}^{-1}(z), \mathbf{w} \in \mathbb{P}^{-1}(w).$$

The holomorphic isometry group of $\mathbf{H}_{\mathbb{H}}^n$ with respect to the Bergman metric is the projective unitary group $\mathrm{PSp}(n, 1) = \mathrm{Sp}(n, 1) / \pm I_{n+1}$ and acts on $\mathbb{P}(\mathbb{H}^{n,1})$ by matrix multiplication.

Let

$$\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n, \mathbb{H}) : A A^* = I_n\}.$$

Since the stabilizer of the point of $\mathbf{H}_{\mathbb{H}}^n$ with the homogeneous coordinates $(0, \dots, 0, 1)$ is

$$\mathrm{Sp}(n) \times \mathrm{Sp}(1) = \left\{ \begin{pmatrix} A & 0 \\ 0 & q \end{pmatrix} : A \in \mathrm{Sp}(n), q \in \mathrm{Sp}(1) \right\},$$

we have the following identification

$$\mathbf{H}_{\mathbb{H}}^n = \mathrm{Sp}(n, 1) / \mathrm{Sp}(n) \times \mathrm{Sp}(1). \quad (2)$$

3 The Lie group $\mathrm{Sp}(n, 1)$

This section contains some necessary materials of the Lie group $\mathrm{Sp}(n, 1)$ including the Cartan decomposition and the standard \mathbb{R} -vector space basis of the Lie algebra $\mathfrak{sp}(n, 1)$, and the canonical metric in Lie group $\mathrm{Sp}(n, 1)$.

3.1 The Cartan decomposition of $\mathfrak{sp}(n, 1)$

A matrix Lie group is a closed subgroup of $\mathrm{GL}(n, \mathbb{H})$. Recall that for a square matrix X ,

$$e^X = I + X + \frac{1}{2}X^2 + \cdots.$$

The Lie algebra of a matrix Lie group G is a vector space, defined as the set of matrices X such that $e^{tX} \in G$, for all real numbers t . The Lie algebra of $\mathrm{GL}(n, \mathbb{H})$, denoted by $\mathfrak{gl}(n, \mathbb{H})$, is the set of $n \times n$ matrices over \mathbb{H} .

The Lie algebra of $\mathrm{Sp}(n, 1)$ is defined and denoted by

$$\mathfrak{sp}(n, 1) = \{X \in \mathfrak{gl}(n+1, \mathbb{H}) : JX^*J = -X\}.$$

The fixed point set of the Cartan involution $\theta(X) = JXJ$ is a maximal compact subgroup K of $\mathrm{Sp}(n, 1)$ isomorphic to $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. The corresponding standard Cartan decomposition

$$\mathfrak{sp}(n, 1) = \mathfrak{k} + \mathfrak{p}$$

is given by

$$\begin{aligned} \mathfrak{k} &= \left\{ \begin{pmatrix} M & 0 \\ 0 & q \end{pmatrix} : M \in \mathfrak{sp}(n), q \in \mathfrak{sp}(1) \right\}, \\ \mathfrak{p} &= \left\{ \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix} : Z \in \mathbb{H}^n \right\}, \end{aligned}$$

where

$$\mathfrak{sp}(n) = \{M \in \mathfrak{gl}(n, \mathbb{H}) : M + M^* = 0\}.$$

The Lie bracket of a matrix Lie algebra is determined by matrix operations

$$[X, Y] = XY - YX.$$

Definition 3.1. For each n , let $e_{jk} \in \mathfrak{gl}(n+1, \mathbb{H})$ be the matrix with 1 in the jk -position and 0 elsewhere. We define

$$\alpha_{jk} = (e_{jk} - e_{kj}), \quad \beta_{jk} = (e_{jk} + e_{kj}). \quad (3)$$

The following proposition describes the Lie bracket of $\mathfrak{sp}(n, 1)$. The proof involves straightforward calculation from the fact

$$e_{ij}e_{kl} = \delta_{jk}e_{il}$$

and the definitions of α_{ij} and β_{ij} and therefore is omitted.

Proposition 3.1. (cf. [3, Proposition 2.2]) Let $\mathbf{I}_1 = \mathbf{i}, \mathbf{I}_2 = \mathbf{j}$ and $\mathbf{I}_3 = \mathbf{k}$. For $1 \leq j < k \leq n, 1 \leq l < m \leq n$, we have the following equalities:

$$[\alpha_{jk}, \alpha_{lm}] = \delta_{kl}\alpha_{jm} + \delta_{km}\alpha_{lj} + \delta_{jm}\alpha_{kl} + \delta_{lj}\alpha_{mk}, \quad (4)$$

$$[\alpha_{jk}, \mathbf{I}_t\beta_{lm}] = \mathbf{I}_t(\delta_{kl}\beta_{jm} + \delta_{km}\beta_{jl} - \delta_{jm}\beta_{kl} - \delta_{lj}\beta_{km}), t = 1, 2, 3, \quad (5)$$

$$[\alpha_{jk}, \mathbf{I}_te_{ii}] = \mathbf{I}_t(\delta_{ki}\beta_{ji} - \delta_{ji}\beta_{ki}), t = 1, 2, 3, \quad (6)$$

$$[\alpha_{jk}, \beta_{l,n+1}] = \delta_{lk}\beta_{j,n+1} - \delta_{jl}\beta_{k,n+1}, \quad (7)$$

$$[\alpha_{jk}, \mathbf{I}_t\alpha_{l,n+1}] = \mathbf{I}_t(\delta_{lk}\alpha_{j,n+1} - \delta_{lj}\alpha_{k,n+1}), t = 1, 2, 3, \quad (8)$$

$$[\mathbf{I}_t\beta_{jk}, \mathbf{I}_t\beta_{lm}] = -(\delta_{kl}\alpha_{jm} + \delta_{km}\alpha_{jl} + \delta_{jm}\alpha_{kl} + \delta_{lj}\alpha_{km}), t = 1, 2, 3, \quad (9)$$

$$[\mathbf{I}_t\beta_{jk}, \mathbf{I}_s\beta_{lm}] = \mathbf{I}_t\mathbf{I}_s(\delta_{kl}\beta_{jm} + \delta_{km}\beta_{jl} + \delta_{jm}\beta_{kl} + \delta_{lj}\beta_{km}), t \neq s, \quad (10)$$

$$[\mathbf{I}_t\beta_{jk}, \mathbf{I}_te_{ii}] = -(\delta_{ki}\alpha_{ji} + \delta_{ji}\alpha_{ki}), t = 1, 2, 3, \quad (11)$$

$$[\mathbf{I}_t\beta_{jk}, \mathbf{I}_se_{ii}] = \mathbf{I}_t\mathbf{I}_s(\delta_{ki}\beta_{ji} + \delta_{ji}\beta_{ki}), t \neq s, \quad (12)$$

$$[\mathbf{I}_t\beta_{jk}, \beta_{l,n+1}] = \mathbf{I}_t(\delta_{lk}\alpha_{j,n+1} + \delta_{jl}\alpha_{k,n+1}), t = 1, 2, 3, \quad (13)$$

$$[\mathbf{I}_t\beta_{jk}, \mathbf{I}_t\alpha_{l,n+1}] = -(\delta_{lk}\beta_{j,n+1} + \delta_{jl}\beta_{k,n+1}), t = 1, 2, 3, \quad (14)$$

$$[\mathbf{I}_t\beta_{jk}, \mathbf{I}_s\alpha_{l,n+1}] = \mathbf{I}_t\mathbf{I}_s(\delta_{lk}\alpha_{j,n+1} + \delta_{jl}\alpha_{k,n+1}), t \neq s, \quad (15)$$

$$[\mathbf{I}_te_{ii}, \mathbf{I}_te_{ii}] = 0, t = 1, 2, 3, \quad (16)$$

$$[\mathbf{I}_te_{ii}, \mathbf{I}_se_{ii}] = 2\mathbf{I}_t\mathbf{I}_se_{ii}, t \neq s, \quad (17)$$

$$[\mathbf{I}_te_{ii}, \beta_{j,n+1}] = \mathbf{I}_t(\delta_{ij}\alpha_{i,n+1} + \delta_{i,n+1}\alpha_{ij}), t = 1, 2, 3, \quad (18)$$

$$[\mathbf{I}_te_{ii}, \mathbf{I}_t\alpha_{j,n+1}] = -(\delta_{ij}\beta_{i,n+1} - \delta_{i,n+1}\beta_{ij}), t = 1, 2, 3, \quad (19)$$

$$[\mathbf{I}_te_{ii}, \mathbf{I}_s\alpha_{j,n+1}] = \mathbf{I}_t\mathbf{I}_s(\delta_{ij}\alpha_{i,n+1} - \delta_{i,n+1}\alpha_{ij}), t \neq s, \quad (20)$$

$$[\beta_{j,n+1}, \beta_{k,n+1}] = \alpha_{jk}, \quad (21)$$

$$[\beta_{j,n+1}, \mathbf{I}_t\alpha_{k,n+1}] = \mathbf{I}_t(-\beta_{jk} + 2\delta_{jk}e_{n+1,n+1}), t = 1, 2, 3, \quad (22)$$

$$[\mathbf{I}_t\alpha_{j,n+1}, \mathbf{I}_t\alpha_{k,n+1}] = \alpha_{jk}, t = 1, 2, 3, \quad (23)$$

$$[\mathbf{I}_t\alpha_{j,n+1}, \mathbf{I}_s\alpha_{k,n+1}] = \mathbf{I}_s\mathbf{I}_t(\beta_{jk} + 2\delta_{jk}e_{n+1,n+1}), t \neq s. \quad (24)$$

By the above proposition, we can verify the following proposition.

Proposition 3.2. *The Cartan decomposition $\mathfrak{sp}(n, 1) = \mathfrak{k} \oplus \mathfrak{p}$ have the following properties:*

$$\mathfrak{k} = \text{span}\{\alpha_{jk}, \mathbf{i}\beta_{jk}, \mathbf{j}\beta_{jk}, \mathbf{k}\beta_{jk}, 1 \leq j < k \leq n, \mathbf{i}e_{ii}, \mathbf{j}e_{ii}, \mathbf{k}e_{ii}, i = 1, 2 \cdots n+1\}, \quad (25)$$

$$\mathfrak{p} = \text{span}\{\beta_{j,n+1}, \mathbf{i}\alpha_{j,n+1}, \mathbf{j}\alpha_{j,n+1}, \mathbf{k}\alpha_{j,n+1}, 1 \leq j \leq n\}, \quad (26)$$

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \quad (27)$$

3.2 The Canonical Metric of $\text{Sp}(n, 1)$

Definition 3.2. *The standard \mathbb{R} -vector space basis for $\mathfrak{sp}(n, 1)$, denoted by \mathfrak{B} , consists of the following set of $2n^2 + 5n + 3$ matrices:*

- $\alpha_{jk}, \mathbf{i}\beta_{jk}, \mathbf{j}\beta_{jk}, \mathbf{k}\beta_{jk}, 1 \leq j < k \leq n$; there are $2n^2 - 2n$ of these.
- $\sqrt{2}\mathbf{i}e_{ii}, \sqrt{2}\mathbf{j}e_{ii}, \sqrt{2}\mathbf{k}e_{ii}, i = 1, 2 \cdots n+1$; there are $3n + 3$ of these.
- $\beta_{j,n+1}, \mathbf{i}\alpha_{j,n+1}, \mathbf{j}\alpha_{j,n+1}, \mathbf{k}\alpha_{j,n+1}, 1 \leq j \leq n$; there are $4n$ of these.

For $X \in \mathfrak{sp}(n, 1)$, the adjoint action of X is the $\mathfrak{sp}(n, 1)$ -endomorphism defined by the Lie bracket

$$\text{ad}X(Y) = [X, Y].$$

We relabel the above standard basis of $\mathfrak{sp}(n, 1)$ according to the order of sequence as $e_1, \dots, e_{2n^2+5n+3}$. Let $C_{ij} \in \mathbb{R}^{2n^2+5n+3}$ be the coefficients of $\text{ade}_i(e_j)$ represented by the basis. That is

$$\text{ad}e_i(e_j) = [e_i, e_j] = (e_1, e_2, \dots, e_{2n^2+5n+3})C_{ij}. \quad (28)$$

We mention that C_{ij} can be read off from Proposition 3.1.

Let

$$X = \sum_{i=1}^{2n^2+5n+3} x_i e_i, \quad x_i \in \mathbb{R}.$$

Then

$$\text{ad}X = \left(\sum_{i=1}^{2n^2+5n+3} x_i C_{i1}, \dots, \sum_{i=1}^{2n^2+5n+3} x_i C_{i,2n^2+5n+3} \right). \quad (29)$$

We note that $\text{ad}X$ is a real square matrix of dimension $2n^2 + 5n + 3$.

The Killing form on $\mathfrak{sp}(n, 1)$ is a symmetric bilinear form given by

$$B(X, Y) = \text{trace}(\text{ad}X \text{ad}Y).$$

Let

$$Y = \sum_{i=1}^{2n^2+5n+3} y_i e_i, \quad y_i \in \mathbb{R}.$$

Then

$$B(X, Y) = -8(n+2) \sum_{i=1}^{2n^2+n+3} x_i y_i + 8(n+2) \sum_{i=2n^2+n+4}^{2n^2+5n+3} x_i y_i. \quad (30)$$

The Killing form enjoys the following important property:

$$B([X, Y], Z) + B(Y, [X, Z]) = 0, \quad \text{for } X, Y, Z \in \mathfrak{sp}(n, 1). \quad (31)$$

A positive definite inner product on $\mathfrak{sp}(n, 1)$ is defined by

$$\langle X, Y \rangle = \begin{cases} B(X, Y) & \text{for } X, Y \in \mathfrak{p}, \\ -B(X, Y) & \text{for } X, Y \in \mathfrak{k}, \\ 0 & \text{otherwise.} \end{cases} \quad (32)$$

By identifying $\mathfrak{sp}(n, 1)$ with the tangent space at the identity of $\text{Sp}(n, 1)$, we can extend $\langle \cdot, \cdot \rangle$ to a left invariant Riemannian metric over $\text{Sp}(n, 1)$. We denote this metric by g and refer to it as the canonical metric for $\text{Sp}(n, 1)$.

By (30) and (32), we have the following lemma.

Lemma 3.1. *For $X, Y \in \mathfrak{B}$,*

$$\langle X, Y \rangle = \begin{cases} 8(n+2) & X = Y \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.1. *The matrix representation for the canonical metric g of $\text{Sp}(n, 1)$ is the square $2n^2 + 5n + 3$ diagonal matrix*

$$\begin{pmatrix} 8(n+2) & & & \\ & 8(n+2) & & \\ & & \ddots & \\ & & & 8(n+2) \end{pmatrix}. \quad (33)$$

Definition 3.3. *Let g be the canonical metric on $\text{Sp}(n, 1)$. The metric \tilde{g} on $\text{Sp}(n, 1)$ is defined by*

$$\tilde{g} = \frac{1}{2(n+2)} g. \quad (34)$$

We will show in Section 5 that the metric \tilde{g} on $\mathrm{Sp}(n, 1)$ induces holomorphic sectional curvature -1 on the quotient $\mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$.

The canonical metric g on a Lie algebra \mathfrak{g} induces a norm given by

$$\|X\| = \langle X, X \rangle^{\frac{1}{2}}.$$

Let

$$N(\mathrm{ad}X) = \sup\{\|\mathrm{ad}X(Y)\| : Y \in \mathfrak{g}, \|Y\| = 1\}, \quad (35)$$

$$C_1 = \sup\{N(\mathrm{ad}X) : X \in \mathfrak{p}, \|X\| = 1\}$$

and

$$C_2 = \sup\{N(\mathrm{ad}U) : U \in \mathfrak{k}, \|U\| = 1\}.$$

The appendix to [20] includes a table of the constants C_1 and C_2 for noncompact and nonexceptional Lie groups. The values for $\mathrm{Sp}(n, 1)$ are

$$C_1 = \frac{1}{\sqrt{2(n+2)}}, \quad C_2 = \sqrt{2}C_1.$$

With respect to the scaled canonical metric \tilde{g} , we have

$$C_1 = 1, \quad C_2 = \sqrt{2}. \quad (36)$$

4 The Sectional Curvature of $\mathrm{Sp}(n, 1)$

This section aims to obtain the bound of the sectional curvatures of $\mathrm{Sp}(n, 1)$ with respect to the scaled canonical metric.

4.1 The connection and curvature

By the fundamental theorem of Riemannian geometry, a connection ∇ on the tangent bundle of a manifold can be expressed in terms of a left invariant metric \langle, \rangle by the *Koszul formula*. For any left invariant vector fields X, Y, Z , we have

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \{ \langle [X, Y], Z \rangle - \langle Y, [X, Z] \rangle - \langle X, [Y, Z] \rangle \}. \quad (37)$$

The *curvature tensor* of a connection ∇ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (38)$$

We mention that a connection is torsion free

$$\nabla_X Y - \nabla_Y X = [X, Y]. \quad (39)$$

When a Lie group G is semisimple and compact, the canonical metric is the negative of the Killing form and induces a biinvariant metric on G . The connection and curvature can be described in terms of the Lie bracket in a simple way [17, Proposition 12 in Chapter 4].

When G is semisimple and noncompact, a canonical metric is biinvariant only when restricted to K , the maximal compact subgroup of G with Lie algebra \mathfrak{k} . Adeboye and Wei have derived similar formulae for the connection and curvature for this case in [2, Proposition 3.3] and [3, Proposition 2.7].

We mention that in order to obtain better estimate, we will use some formulae of the connection and curvature which are slight different from those in [3]. Those formulae can be easily derived by the properties of (31),(37)-(39) and the Jacobi identity. For the convenience of the readers, we recall them as the following two propositions.

Proposition 4.1. ([2, Lemma 3.2]) *Let $U, V, W \in \mathfrak{k}$ and $X, Y, Z \in \mathfrak{p}$. Then we have the following equalities:*

$$\nabla_U V = \frac{1}{2}[U, V], \quad \nabla_U X = \frac{3}{2}[U, X]; \quad (40)$$

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad \nabla_X U = -\frac{1}{2}[X, U]. \quad (41)$$

Proposition 4.2. (cf. [3, Proposition 2.7]) *Let $U, V, W \in \mathfrak{k}$ and $X, Y, Z \in \mathfrak{p}$. Then we have the following equalities:*

$$R(U, V)W = \frac{1}{4}[[V, U], W], \quad (42)$$

$$R(X, Y)Z = -\frac{7}{4}[[X, Y], Z], \quad (43)$$

$$R(X, V)Y = \frac{1}{4}[X, [V, Y]] + \frac{1}{4}[V, [X, Y]], \quad (44)$$

$$R(X, V)V = \frac{1}{4}[V, [X, V]], \quad (45)$$

$$R(X, Y)V = \frac{3}{4}[V, [X, Y]]. \quad (46)$$

In particular

$$\langle R(U, V)W, X \rangle = 0, \quad (47)$$

$$\langle R(X, Y)Z, U \rangle = 0, \quad (48)$$

$$\langle R(U, V)V, U \rangle = \frac{1}{4}\| [U, V] \|^2, \quad (49)$$

$$\langle R(X, Y)Y, X \rangle = -\frac{7}{4}\| [X, Y] \|^2, \quad (50)$$

$$\langle R(U, X)X, U \rangle = \frac{1}{4}\| [U, X] \|^2. \quad (51)$$

Proof. Comparing with [3, Proposition 2.7]), we only need to prove (44)-(46). Note that

$$-\frac{1}{2}[X, [V, Y]] - \frac{1}{2}[V, [Y, X]] - \frac{1}{2}[Y, [X, V]] = 0.$$

We obtain that

$$\begin{aligned} R(X, V)Y &= \nabla_X \nabla_V Y - \nabla_V \nabla_X Y - \nabla_{[X, V]} Y \\ &= \frac{3}{4}[X, [V, Y]] - \frac{1}{4}[V, [X, Y]] + \frac{1}{2}[Y, [X, V]] \\ &= \frac{1}{4}[X, [V, Y]] + \frac{1}{4}[V, [X, Y]]. \end{aligned}$$

Similarly we have

$$\begin{aligned} R(X, V)V &= \nabla_X \nabla_V V - \nabla_V \nabla_X V - \nabla_{[X, V]} V \\ &= \frac{3}{4}[V, [X, V]] + \frac{1}{2}[[X, V], V] \\ &= \frac{1}{4}[V, [X, V]] \end{aligned}$$

and

$$\begin{aligned}
R(X, Y)V &= \nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X, Y]} V \\
&= -\frac{1}{4}[X, [Y, V]] + \frac{1}{4}[Y, [X, V]] - \frac{1}{2}[[X, Y], V] \\
&= \frac{3}{4}[V, [X, Y]].
\end{aligned}$$

□

Definition 4.1. The sectional curvature of the planes spanned by $X, Y \in \mathfrak{g}$ is denoted and defined by

$$K(X, Y) = \frac{\langle R(X, Y)Y, X \rangle}{\|X\|^2\|Y\|^2 - \langle X, Y \rangle^2}. \quad (52)$$

Proposition 4.3. The sectional curvature of $\mathrm{Sp}(n, 1)$ with respect to the metric \tilde{g} at the planes spanned by standard basis elements is bounded above by $\frac{1}{2}$.

Proof. Since the basis elements are mutually orthogonal, the sectional curvature at the plane spanned by any distinct elements $X, Y \in \mathfrak{B}$ is given by

$$K(X, Y) = \frac{\langle R(X, Y)Y, X \rangle}{\|X\|^2\|Y\|^2}.$$

By (49)-(51) and Proposition 3.1, the largest sectional curvature spanned by basis directions are the planes spanned by $\mathbf{i}\sqrt{2}e_{ii}, \mathbf{j}\sqrt{2}e_{ii}; \mathbf{i}\sqrt{2}e_{ii}, \mathbf{k}\sqrt{2}e_{ii}$ or $\mathbf{j}\sqrt{2}e_{ii}, \mathbf{k}\sqrt{2}e_{ii}$. The largest sectional curvature is given by

$$K(\mathbf{i}\sqrt{2}e_{ii}, \mathbf{j}\sqrt{2}e_{ii}) = \frac{\frac{1}{4}\|\mathbf{i}\sqrt{2}e_{ii}, \mathbf{j}\sqrt{2}e_{ii}\|^2}{\|\mathbf{i}\sqrt{2}e_{ii}\|^2\|\mathbf{j}\sqrt{2}e_{ii}\|^2} = \frac{1}{4} \frac{\|4\mathbf{k}e_{ii}\|^2}{16} = \frac{1}{2}. \quad (53)$$

□

Let

$$R(X_1, X_2, X_3, X_4) := \langle R(X_1, X_2)X_3, X_4 \rangle.$$

We recall the following facts [17, Page 33] for any left invariant vector fields X_1, X_2, X_3, X_4 :

$$R(X_1, X_2, X_3, X_4) = -R(X_2, X_1, X_3, X_4) = R(X_2, X_1, X_4, X_3), \quad (54)$$

$$R(X_1, X_2, X_3, X_4) = R(X_3, X_4, X_1, X_2). \quad (55)$$

Proposition 4.4. The sectional curvatures of $\mathrm{Sp}(n, 1)$ with respect to \tilde{g} are bounded above by $\frac{3+4\sqrt{2}}{2}$.

Proof. Let $U, V \in \mathfrak{k}$ and $X, Y \in \mathfrak{p}$. Then by the above properties (54), (55) and (47), (48) we can reduce the sixteen items of the expansion of $\langle R(X + U, Y + V)(Y + V), X + U \rangle$ to eight items. That is

$$\begin{aligned}
\langle R(X + U, Y + V)(Y + V), X + U \rangle &= \langle R(X, Y)Y, X \rangle + \langle R(U, V)V, U \rangle + \langle R(U, Y)Y, U \rangle \\
&\quad + \langle R(X, V)V, X \rangle + 2\langle R(X, Y)V, U \rangle + 2\langle R(X, V)Y, U \rangle.
\end{aligned}$$

Assume that $\|U + X\| = 1, \|V + Y\| = 1$ and $\langle U + X, V + Y \rangle = 0$. Let c_V, c_U, c_X, c_Y be real numbers satisfying

$$\|c_V V\| = \|c_U U\| = \|c_X X\| = \|c_Y Y\| = 1. \quad (56)$$

By our assumption, it obvious that

$$|c_V|, |c_U|, |c_X|, |c_Y| \geq 1.$$

It follows from (35) that

$$\|\text{ad}V(U)\| = \|[V, U]\| \leq \|[c_V V, c_U U]\| = \|\text{ad}c_V V(c_U U)\| \leq C_2.$$

Similarly we have

$$\|\text{ad}V(X)\| \leq C_2, \|\text{ad}X(U)\| \leq C_1, \|\text{ad}X(Y)\| \leq C_1.$$

Therefore by (49), (51), (45) and (36) we have

$$\langle R(U, V)V, U \rangle = \frac{1}{4}\|\text{ad}V(U)\|^2 \leq \frac{1}{4}C_2^2 = \frac{1}{2},$$

$$\langle R(U, Y)Y, U \rangle = \frac{1}{4}\|\text{ad}Y(U)\|^2 \leq \frac{1}{4}C_1^2 = \frac{1}{4}$$

and

$$\langle R(X, V)V, X \rangle = \langle \frac{1}{4}[V, [X, V]], X \rangle = \frac{1}{4}\|\text{ad}X(V)\|^2 \leq \frac{1}{4}C_1^2 = \frac{1}{4}.$$

By (46) we have

$$\begin{aligned} \langle R(X, Y)V, U \rangle &= \frac{3}{4}\langle [X, Y], [U, V] \rangle \\ &\leq \frac{3}{4}\|[X, Y]\| \|[U, V]\| \\ &= \frac{3}{4}\|\text{ad}X(Y)\| \|\text{ad}U(V)\| \\ &\leq \frac{3C_1C_2}{4} = \frac{3\sqrt{2}}{4}. \end{aligned}$$

By (44) we have

$$\begin{aligned} \langle R(X, V)Y, U \rangle &= \frac{1}{4}\langle [V, Y], [X, U] \rangle + \frac{1}{4}\langle [X, Y], [U, V] \rangle \\ &\leq \frac{1}{4}\left[\|[X, U]\| \|[Y, V]\| + \frac{1}{4}\|[X, Y]\| \|[U, V]\| \right] \\ &= \frac{1}{4}\left[\|\text{ad}X(U)\| \|\text{ad}Y(V)\| + \frac{1}{4}\|\text{ad}X(Y)\| \|\text{ad}U(V)\| \right] \\ &\leq \frac{1}{4}(C_1^2 + C_1C_2) = \frac{1 + \sqrt{2}}{4}. \end{aligned}$$

Noting that

$$\langle R(X, Y)Y, X \rangle = -\frac{7}{4}\|[X, Y]\|^2 \leq 0,$$

we obtain that the sectional curvatures of $\text{Sp}(n, 1)$ with respect to \tilde{g} are bounded above by

$$\frac{1}{2} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1 + \sqrt{2}}{4} + 2 \cdot \frac{3\sqrt{2}}{4} = \frac{3 + 4\sqrt{2}}{2}.$$

□

5 The volume of quaternionic hyperbolic orbifolds

This section contains the proof of Theorem 1.3 with the similar route map employed in [3]. First, we construct a Riemannian submersion from the quotient $\mathrm{Sp}(n, 1)/\Gamma$ to the quotient $\mathbf{H}_{\mathbb{H}}^n/\Gamma$. With this Riemannian submersion, we can employ Wang's result [20, Theorem 5.2] to produce an inscribed ball of radius $\frac{R_{\mathrm{Sp}(n, 1)}}{2}$ in $\mathbf{H}_{\mathbb{H}}^n/\Gamma$ and obtain the lower bound by a comparison theorem of Gunther [9, Theorem 3.101].

5.1 Riemannian Submersions

Definition 5.1. *Let (M, g) and (N, h) be Riemannian manifolds and $q : M \rightarrow N$ a surjective submersion. For each point $x \in M$, the tangent space $T_x M$ can be decomposed into the orthogonal direct sum*

$$T_x M = (\mathrm{Ker} \, dq)_x^\perp + (\mathrm{Ker} \, dq)_x.$$

The map q is said to be a Riemannian submersion if

$$g(X, Y) = h(dqX, dqY), \forall X, Y \in (\mathrm{Ker} \, dq)_x^\perp \text{ for some } x \in M.$$

Let X, Y be orthonormal vector fields on N and let \tilde{X}, \tilde{Y} be their horizontal lifts to M . O'Neill's formula [9, Page 127] relates the sectional curvature of the base space of a Riemannian submersion with that of the total space

$$K_b(X, Y) = K_t(\tilde{X}, \tilde{Y}) + \frac{3}{4} \|[\tilde{X}, \tilde{Y}]^\perp\|^2, \quad (57)$$

where Z^\perp represents the vertical component of Z .

Definition 5.2. *Let \mathbb{J} be the complex structure on \mathfrak{p} such that*

$$\mathbb{J}X = \sum_{j=1}^n (a_{2j}\beta_{j,n+1} - a_{1j}\mathbf{i}\alpha_{j,n+1} - a_{4j}\mathbf{j}\alpha_{j,n+1} + a_{3j}\mathbf{j}\alpha_{j,n+1}), \quad (58)$$

for $X = \sum_{j=1}^n (a_{1j}\beta_{j,n+1} + a_{2j}\mathbf{i}\alpha_{j,n+1} + a_{3j}\mathbf{j}\alpha_{j,n+1} + a_{4j}\mathbf{k}\alpha_{j,n+1}) \in \mathfrak{p}$.

We remind that

$$B(\mathbb{J}X, \mathbb{J}Y) = B(X, Y).$$

This implies that the complex structure preserves the Killing form $B(X, Y)$.

Proposition 5.1. *Let*

$$X = \sum_{j=1}^n (a_{1j}\beta_{j,n+1} + a_{2j}\mathbf{i}\alpha_{j,n+1} + a_{3j}\mathbf{j}\alpha_{j,n+1} + a_{4j}\mathbf{k}\alpha_{j,n+1}) \quad (59)$$

and

$$Y = \mathbb{J}X = \sum_{k=1}^n (a_{2k}\beta_{k,n+1} - a_{1k}\mathbf{i}\alpha_{k,n+1} - a_{4k}\mathbf{j}\alpha_{k,n+1} + a_{3k}\mathbf{k}\alpha_{k,n+1}), \quad (60)$$

where

$$\sum_{j=1}^n (a_{1j}^2 + a_{2j}^2 + a_{3j}^2 + a_{4j}^2) = \frac{1}{4}.$$

Then

$$\|[X, Y]\|^2 = 1.$$

Proof. It is obvious that $\|X\| = 1$, $\|Y\| = 1$ and $\langle X, Y \rangle = 0$. By Proposition 3.1 we have

$$\begin{aligned}
[X, Y] &= \sum_{j \neq k} \left\{ (a_{1j}a_{2k} - a_{2j}a_{1k} - a_{3j}a_{4k} + a_{4j}a_{3k})\alpha_{jk} \right. \\
&\quad + (a_{1j}a_{1k} + a_{2j}a_{2k} - a_{3j}a_{3k} - a_{4j}a_{4k})\mathbf{i}\beta_{jk} \\
&\quad + (a_{1j}a_{4k} + a_{2j}a_{3k} + a_{3j}a_{2k} + a_{4j}a_{1k})\mathbf{j}\beta_{jk} \\
&\quad + \left. (-a_{1j}a_{3k} + a_{2j}a_{4k} - a_{3j}a_{1k} + a_{4j}a_{2k})\mathbf{k}\beta_{jk} \right\} \\
&\quad + \sum_{j=1}^n \left\{ \sqrt{2}(a_{1j}^2 + a_{2j}^2 - a_{3j}^2 - a_{4j}^2)\sqrt{2}\mathbf{i}e_{jj} \right. \\
&\quad + 2\sqrt{2}(a_{1j}a_{4j} + a_{2j}a_{3j})\sqrt{2}\mathbf{j}e_{jj} \\
&\quad + 2\sqrt{2}(a_{2j}a_{4j} - a_{1j}a_{3j})\sqrt{2}\mathbf{k}e_{jj} \\
&\quad + \left. \sqrt{2}(a_{1j}^2 + a_{2j}^2 + a_{3j}^2 + a_{4j}^2)\sqrt{2}\mathbf{i}e_{n+1,n+1} \right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\|[X, Y]\|^2}{4} &= 4 \sum_{j < k} \left\{ (a_{1j}a_{2k} - a_{2j}a_{1k} - a_{3j}a_{4k} + a_{4j}a_{3k})^2 \right. \\
&\quad + (a_{1j}a_{1k} + a_{2j}a_{2k} - a_{3j}a_{3k} - a_{4j}a_{4k})^2 \\
&\quad + (a_{1j}a_{4k} + a_{2j}a_{3k} + a_{3j}a_{2k} + a_{4j}a_{1k})^2 \\
&\quad + \left. (-a_{1j}a_{3k} + a_{2j}a_{4k} - a_{3j}a_{1k} + a_{4j}a_{2k})^2 \right\} \\
&\quad + \sum_{j=1}^n \left\{ 2(a_{1j}^2 + a_{2j}^2 - a_{3j}^2 - a_{4j}^2)^2 + 8(a_{1j}a_{4j} + a_{2j}a_{3j})^2 + 8(a_{2j}a_{4j} - a_{1j}a_{3j})^2 \right\} + \frac{1}{8} \\
&= 4 \sum_{j < k} (a_{1j}^2 + a_{2j}^2 + a_{3j}^2 + a_{4j}^2)(a_{1k}^2 + a_{2k}^2 + a_{3k}^2 + a_{4k}^2) + \sum_{j=1}^n \left\{ 2(a_{1j}^2 + a_{2j}^2 + a_{3j}^2 + a_{4j}^2)^2 \right\} + \frac{1}{8} \\
&= 2 \left(\sum_{j=1}^n (a_{1j}^2 + a_{2j}^2 + a_{3j}^2 + a_{4j}^2) \right)^2 + \frac{1}{8} = \frac{1}{4}.
\end{aligned}$$

□

Proposition 5.2. *Consider the quotient map*

$$\pi : \mathrm{Sp}(n, 1) \rightarrow \mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1). \quad (61)$$

Then the restriction of the inner product $\langle X, Y \rangle$, defined on $\mathfrak{sp}(n, 1) = \mathfrak{k} \oplus \mathfrak{p}$, to

$$d_e\pi(\mathfrak{p}) = T_{\pi(e)}\mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1),$$

induces a Riemannian metric on the quotient space. That is the map π is a Riemannian submersion.

Proof. We need to show that $\mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ has constant holomorphic sectional curvature -1 with the restriction of the scaled canonical metric \tilde{g} .

Let X represent both a unit vector field on $\mathrm{Sp}(n, 1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ as well as its horizontal lift. Let X and $Y = \mathbb{J}X$ be given by (59) and (60). By (52) we have

$$K_t(X, Y) = \langle R(X, Y)Y, X \rangle = -\frac{7}{4}\|[X, Y]\|^2.$$

Since $[X, Y] \in \mathfrak{k}$, $[X, Y]^\perp = [X, Y]$. It follows from Proposition 5.1 and O'Neill's formula (57) that

$$K_b(X, \mathbb{J}X) = K_b(X, Y) = -\|[X, Y]\|^2 = -1.$$

This implies that π is a Riemannian submersion from $\mathrm{Sp}(n, 1)$ to the quaternionic hyperbolic n -space $\mathbf{H}_{\mathbb{H}}^n$. \square

5.2 Wang and Gunther's results

The following result gives Wang's quantitative version of the well-known result of Kazhdan-Marglis [13].

Lemma 5.1. ([20, Theorem 5.2]) *Let G be a semisimple Lie group without compact factor, let id be the identity of G , let ρ be the distance function derived from a canonical metric, and let*

$$B_G = \{x \in G : \rho(id, x) \leq R_G\}.$$

Then for any discrete subgroup Γ of G , there exists $g \in G$ such that $B_G \cap g\Gamma g^{-1} = id$.

Wang also showed that number R_G is less than the injectivity radius of G . Consequently, the volume of the fundamental domain of any discrete subgroup Γ of G , when viewed as a group of left translations of G , is bounded from below by the volume of a ρ -ball of radius $\frac{R_G}{2}$.

Since $\mathrm{Sp}(n, 1)$ is a semisimple Lie group without compact factor. Let C_1 and C_2 be given by (36). By [20] the number $R_{\mathrm{Sp}(n, 1)}$ is the least positive zero of the real-valued function

$$F(t) = \exp C_1 t - 2 \sin C_2 t - \frac{C_1 t}{\exp C_1 t - 1}. \quad (62)$$

That is

$$R_{\mathrm{Sp}(n, 1)} \approx 0.228 \dots \quad (63)$$

Let $V(d, k, r)$ denote the volume of a ball of radius r in the complete simply connected Riemannian manifold of dimension d with constant curvature k . In [2], Adeboye and Wei obtained the following formula

$$V(d, k, r) = \frac{2(\frac{\pi}{k})^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^{\min(r\sqrt{k}, \pi)} \sin^{d-1} \rho d\rho. \quad (64)$$

We recall the following comparison theorem of Gunther.

Lemma 5.2. ([9, Theorem 3.101]) *Let M be a complete Riemannian manifold of dimension d . For $m \in M$, let $B_m(r)$ be a ball which does not meet the cut-locus of m . If the sectional curvatures of M are bounded above by a constant b , then*

$$\mathrm{Vol}[B_m(r)] \geq V(d, b, r).$$

5.3 The proof of main result

In order to prove our main result, we need the following four lemmas. The following two lemmas have been proved in [2, 3].

Lemma 5.3. ([3, Lemma 3.4]) *Let G be a semisimple Lie group and \mathfrak{g} be its Lie algebra, with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let K be the maximal compact subgroup of G with Lie algebra \mathfrak{k} . Then, with respect to the canonical metric, K is totally geodesic in G .*

Lemma 5.4. ([3, Lemma 3.5]) *Let $K \rightarrow M \xrightarrow{q} N$ denote a Riemannian submersion and K is a compact and totally geodesic submanifold of M . Then for any subset $Z \subset N$,*

$$\text{Vol}[q^{-1}(Z)] = \text{Vol}[Z] \cdot \text{Vol}[K].$$

The following simple lemma is the main tool we use to produce our estimate.

Lemma 5.5. *Let Γ be a discrete subgroup of $\text{Sp}(n, 1)$, then*

$$\text{Vol}[\text{Sp}(n, 1)/\Gamma] \geq V(d_0, k_0, r_0),$$

where $d_0 = 2n^2 + 5n + 3$, $k_0 = \frac{3+4\sqrt{2}}{2}$ and $r_0 = 0.114$.

Proof. The inequality follows from Lemma 5.1 and 5.2. The values of d_0, k_0 and r_0 follow from Definition 3.2, Proposition 4.4 and (63), respectively. \square

Lemma 5.6. *With respect to the metric \tilde{g} given by (34),*

$$\text{Vol}[\text{Sp}(n) \times \text{Sp}(1)] = \frac{2^n (\pi)^{(n^2+n+\frac{3}{2})} \Gamma(\frac{4n+1}{2})}{(2n+1)!(2n-1)!(2n-3)! \cdots 5!3!1!}.$$

Proof. The volumes of the classical compact groups are given explicitly in [11, Chapter 9]. The volume formulae with respect to the metric \tilde{g} are

$$\text{Vol}[\text{Sp}(n+1)] = \frac{2^{n+1} (\pi)^{(n+1)(n+2)}}{(2n+1)!(2n-1)!(2n-3)! \cdots 5!3!1!}$$

and

$$\text{Vol}[\text{Sp}(n+1)/\text{Sp}(n) \times \text{Sp}(1)] = \frac{2 (\pi)^{\frac{4n+1}{2}}}{\Gamma(\frac{4n+1}{2})}.$$

Hence

$$\text{Vol}[\text{Sp}(n) \times \text{Sp}(1)] = \frac{2^n (\pi)^{(n^2+n+\frac{3}{2})} \Gamma(\frac{4n+1}{2})}{(2n+1)!(2n-1)!(2n-3)! \cdots 5!3!1!}.$$

\square

We now are ready to give a proof of Theorem 1.3, which for convenience is restated below.

Theorem 1.3 *The volume of a quaternionic hyperbolic n -orbifold is bounded below by $\mathcal{Q}(n)$, an explicit constant depending only on dimension, given by*

$$\mathcal{Q}(n) = \frac{\pi^{\frac{3n}{2}} (2n+1)!(2n-1)! \cdots 5!3!1!}{2^{n-1} \Gamma(\frac{2n^2+5n+3}{2}) \Gamma(\frac{4n+1}{2}) (\frac{3+4\sqrt{2}}{2})^{\frac{2n^2+5n+3}{2}}} \int_0^{0.2372} (\sin \rho)^{2n^2+5n+2} d\rho.$$

Proof. By Proposition 5.2, the holomorphic sectional curvature of $\mathbf{H}_{\mathbb{H}}^n$ is normalized to be -1 by the metric \tilde{g} given by (34). Let Q be a quaternionic hyperbolic n -orbifold given by

$$Q = \mathbf{H}_{\mathbb{H}}^n / \Gamma = [\text{Sp}(n, 1) / \text{Sp}(n) \times \text{Sp}(1)] / \Gamma.$$

Then the quotient map π given by (61) induces another Riemannian submersion

$$\pi' : \text{Sp}(n, 1) / \Gamma \rightarrow Q.$$

The fibers of π' on the smooth points of Q are totally geodesic embedded copies of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. By Lemmas 5.3, 5.4 and 5.5, we have

$$V(d_0, k_0, r_0) \leq \mathrm{Vol}[\mathrm{Sp}(n, 1)/\Gamma] \leq \mathrm{Vol}[\pi^{-1}(Q)] = \mathrm{Vol}[Q] \cdot \mathrm{Vol}[\mathrm{Sp}(n) \times \mathrm{Sp}(1)].$$

Hence

$$\mathrm{Vol}[Q] \geq \frac{V(d_0, k_0, r_0)}{\mathrm{Vol}[\mathrm{Sp}(n) \times \mathrm{Sp}(1)]} = \mathcal{Q}(n).$$

The proof follows from Lemma 5.6 and (64). \square

6 Slight improvement for real and complex cases

By similar way of Proposition 4.4, we can reestimate the sectional curvatures of $\mathrm{SO}_o(n, 1)$ and $\mathrm{SU}(n, 1)$.

Proposition 6.1. (1) The sectional curvatures k of $\mathrm{SO}_o(n, 1)$ with respect to \tilde{g} given by [2, Definition 1.5]

$$\text{are bounded above by } \begin{cases} \frac{1}{4}, & \text{when } n = 2 \text{ (see [2])}; \\ \frac{13}{4}, & \text{when } n = 3; \\ \frac{3+4\sqrt{2}}{2}, & \text{when } n \geq 4. \end{cases}$$

(2) The sectional curvatures of $\mathrm{SU}(n, 1)$ with respect to \tilde{g} given by Definition 2.6 in [3] are bounded above by $\frac{13}{4}$.

In light of the above proposition, we can slight improve the main result Theorem 0.1 in [2, 3] as followings.

Theorem 6.1. (1) The volume of a real hyperbolic n -orbifold is bounded below by $\mathcal{R}(n)$, an explicit constant depending only on dimension, given by

$$\mathcal{R}(n) = \frac{2^{\frac{6-n}{4}} \pi^{\frac{n}{4}} (n-2)!(n-4)! \cdots 1}{(3+4\sqrt{2})^{\frac{n^2+n}{4}} \Gamma(\frac{n^2+n}{4})} \int_0^{0.2372} (\sin \rho)^{\frac{n^2+n-2}{2}} d\rho.$$

(2) The volume of a complex hyperbolic n -orbifold is bounded below by $\mathcal{C}(n)$, an explicit constant depending only on dimension, given by

$$\mathcal{C}(n) = \frac{2^{n^2+n+1} \pi^{\frac{n}{2}} (n-1)!(n-2)! \cdots 3!2!1!}{(13)^{\frac{n^2+2n}{2}} \Gamma(\frac{n^2+2n}{2})} \int_0^{0.2497} (\sin \rho)^{n^2+2n-1} d\rho.$$

The following is a table of the lower bound for the volume of hyperbolic n -orbifolds in [2, 3] and this paper for some cases of $n \leq 4$ (by software Matlab of version R2009b).

	Results in [2,3]		Results of this paper		
n	$\mathcal{R}(n)$	$\mathcal{C}(n)$	$\mathcal{R}(n)$	$\mathcal{C}(n)$	$\mathcal{Q}(n)$
1		0.00168		0.00175	3.6221×10^{-11}
2	0.00125	2.9180×10^{-9}		4.1822×10^{-9}	5.3637×10^{-25}
3	2.4583×10^{-7}	3.6324×10^{-18}	2.8073×10^{-7}	1.1556×10^{-17}	
4	3.1469×10^{-13}	2.2347×10^{-30}	4.0019×10^{-13}	3.7865×10^{-29}	

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